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ON BEHAVIOR OF THE SMALL PERTURBATIONS OF ONE-DIMENSIONAL STEADY TRANSONIC FLOWS*

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Behavior of unsteady perturbations of steady solutions of quasilinear hyperbolic or parabolic degenerate systems of differential equations in partial derivatives is considered in the critical point neighborhood. The sought functions of analyzed equations are assumed dependent on two arguments, viz. coordinate x and time t, with an arbitrary number of sought functions. The point at which one of the system characteristic velocities vanishes, is called critical.

Previously, the development of unsteady perturbations in the critical point neighborhood was studied /1/ on the assumption that the coefficients and the right-hand sides of equations are continuous functions of their arguments. As shown in /1/, the critical points are on such assumptions singular points of a system of ordinary differential equations obtained from the input system for steady solutions, and the behavior of unsteady perturbations in the critical point neighborhood are defined by nonlinear differential equations in partial derivatives of the first order.

Below, the constraints imposed on the right-hand sides of input equations are substantially weakened in that first order discontinuities are admitted in the right-hand sides. Steady and unsteady solutions are considered in the neighborhood of discontinuity points at which simultaneously one of the characteristic velocities changes its sign. At such critical points the derivatives of steady solutions become infinite. A nonlinear differential equation of first order is obtained for the definition of unsteady perturbations whose propagation velocity vanishes in the neighborhood of such critical points. This equation is a generalization of respective equation in /1/ obtained for the case of continuous right-hand sides and differs from it by the presence of a supplementary piecewise constant term. The perturbation whose characteristic velocity vanishes, generates perturbations of other types. Formulas are obtained for the principal part of such perturbations in the critical point proximity.

The obtained results can be used in the analysis of stability of steady solutions of hyperbolic systems of differential equations in the presence of points at which one of the characteristic velocities vanishes. In problems of gasdynamics and physics of gasdynamics the vanishing of a characteristic velocity means that the flow velocity has reached the speed of sound.

1. Let us consider the hyperbolic system of equations for n functions $u_j(x, t)$ dependent on two independent variables, viz. the three-dimensional coordinate x and time t

$$l_j^i(u_k, x)\left[\frac{\partial u_j}{\partial t} + c^i(u_k, x)\frac{\partial u_j}{\partial x}\right] = f^i(u_k, x), \quad (i, j, k = 1, 2, \ldots, n) \tag{1.1}$$

System (1.1) is written in characteristic form, $c^i(u_k, x)$ are the characteristic velocities of the system, and the recurrent subscripts indicate summation from 1 to n.

Owing to the system hyperbolicity matrix (l_j^i) is nondegenerate. Its elements and the characteristic velocities of system (1.1) are assumed to be continuous and differentiable functions of u_k and x with respect to all arguments. Functions $f^i(u_k, x)$ in the right-hand sides of Eqs.(1.1) are considered to be piecewise-continuous, and may have first order discontinuities on some planes $x = \mathrm{const}$ or some surfaces $\vartheta(u_k, x) = 0$ in the space u_k, x. First order partial derivatives of $f^i(u_k, x)$ will be considered as existing and continuous everywhere where $f^i(u_k, x)$ are determinate, except at points belonging to discontinuity surfaces of these functions.

Let in the considered region of variation of variables u_k, x one of the single characteristic velocities of the system of Eqs.(1.1), for example $c^1(u_k, x)$, vanish, while the remaining characteristic velocities $c^\mu(u_k, x)$ $(\mu = 2, \ldots, n)$ are nonzero. We select some steady (time independent) solution $U_j(x)$ $(j = 1, 2, \ldots, n)$ of system (1.1), which intersects surface $c^1(U_k, x) = 0$

*Prikl.Matem.Mekhan.,46,No.6,pp.979-988,1982 at some point x^* and is continuous in its small neighborhood. We call the point of intersection of $U_j(x)$ $(j = 1, 2, \ldots, n)$ with surface $c^1(U_k, x) = 0$ critical, and take it as the three-dimensional coordinate origin. Since we have the freedom of choice of sought functions, we can assume, without loss of generality, that $U_j(0) = 0$ for all j from 1 to n. Then by virtue of the selected origin of coordinate x and of values of quantities U_j, we have $c^1(0, 0, \ldots, 0) = 0$ at the critical point.

In the case of steady solutions system (1.1) becomes the system of ordinary differential equations

$$l_j^i(U_k, x)\, c^i(U_k, x) \frac{dU_j}{dx} = f^i(U_k, x) \tag{1.2}$$

Since by assumption c^1 is the single characteristic velocity, hence when $c^1 = 0$ the first row of the matrix of coefficients vanishes for dU_j/dx and, consequently, the rank of that matrix at points of surface $c^1 = 0$ is equal $n - 1$. At the remaining points of that region this rank is equal n.

If function f^1 is nonzero and continuous at points of surface $c^1 = 0$, the derivatives dU_j/dx $(j = 1, 2, \ldots, n)$ become infinite at these points and change their sign when passing through that surface. This means that a solution which is continuous and single-valued for x exists only in a one-sided neighborhood of the critical point, and that such points can only be considered as one of the boundaries of the interval in which the solution is studied.

If the critical point lies within the interval, then the existence of a solution which is continuous and single-valued for x, function $f^1(U_k\, x)$ must change its sign at the critical point either continuously or discontinuously.

In the first case, points of space U_k, x, at which conditions

$$c^1(U_k, x) = 0, \; f^1(U_k, x) = 0 \tag{1.3}$$

are simultaneously satisfied, are singular points of steady equations (1.2). The steady and unsteady solutions under conditions (1.3) were investigated in /1/.

In the second case, when function f^1 is discontinuous at points of surface $c^1 = 0$, a continuous solution in the two-sided neighborhood of the critical point is possible under conditions that

$$c^1 = 0, \; f_-^1 \neq f_+^1 \tag{1.4}$$

The subscripts minus and plus denote here and subsequently quantities immediately to the left and right of the critical point, respectively.

Under conditions (1.4) the derivatives dU_j/dx becomes infinite at the critical point $x = 0$.

The set of points defined by the equalities (1.3) and the intersection of surface $c^1 = 0$ with the surface where f^1 has a discontinuity constitute $(n-1)$-dimensional surfaces in the $(n+1)$-dimensional space of variables U_k, x.

Note that the requirement for the existence of a continuous steady solution in the critical point neighborhood does not impose any additional conditions on functions f^μ $(\mu = 2, \ldots, n)$. Hence they are considered below, for simplicity, as continuous and differentiable throughout the considered region, including the points where $c^1 = 0$.

2. Let us assume that the steady solution $U_j(x)$ is weakly perturbed, i.e. the solution $u_j(x, t)$ of Eqs. (1.1) is the sum of solutions $U_j(x) + u_j^*(x, t)$, where $u_j^*(x, t)$ is a small unsteady perturbations. Consider the behavior of solution $u_j(x, t)$ of the system of Eqs. (1.1) in the small neighborhood (of dimension δ) of the cirtical point $x = 0$ at which $c^1 = 0$ and $U_j = 0$ $(j = 1, 2, \ldots, n)$.

We consider in this point the case of the discontinuous function f^1 that satisfies conditions (1.4).

We denote by l_{j0}^i the limit values of l_j^i for $x = 0, u_k = 0$, and introduce the new variables

$$\begin{gathered} w_i(x, t) = l_{j0}^i u_j, \; u_j = r_{jk} w_k \\ r_{jk} = (l_{j0}^k)^{-1} \; (i, j, k = 1, 2, \ldots, n) \end{gathered} \tag{2.1}$$

Since the solution of Eqs. (1.1) is considered in the small neighborhood of the critical point, the quantities $u_j(x, t)$ and, consequently, also $w_i(x, t)$ are small. Suppose the quantity $w_1 \equiv w$ is of order $\sqrt{\delta}$, while the quantities $w_\mu(x, t), w_{\mu 0}(t)$, $(\mu = 2, \ldots, n)$, where $w_{\mu 0}(t) = w_\mu(0, t)$, are of order δ. The characteristic time scale quantity is assumed to be of order $\sqrt{\delta}$. These nontrivial and important assumptions will be confirmed subsequently in the case of solutions concentrated close to the critical point under conditions (1.4).

We expand the coefficients and right-hand sides of Eqs. (1.1) in series in w_k and x, retaining in them the principal terms

$$(l_{j0}^1 + l_{j1}^1 w)\left[\frac{\partial u_j}{\partial t} + (c_k^1 w_k + c_x^1 x + c_{11}^1 w^2)\frac{\partial u_j}{\partial x}\right] = f_0^1 + f_1^1 w \tag{2.2}$$

$$(l_{j0}^\mu + l_{j1}^\mu w)\left[\frac{\partial u_j}{\partial t} + (c_0^\mu + c_1^\mu w)\frac{\partial u_j}{\partial x}\right] = f_0^\mu \tag{2.3}$$

$$(i, j, k = 1, 2, \ldots, n,\ \mu = 2, \ldots, n)$$

$$l_{j0}^i = l_j^i(0,0), \quad l_{j1}^i = \frac{\partial l_j^i}{\partial w}(0,0), \quad c_0^\mu = c^\mu(0,0)$$

$$c_k^i = \frac{\partial c^i}{\partial w_k}(0,0), \quad c_{11}^1 = \frac{\partial^2 c^1}{\partial w^2}(0,0), \quad c_x^1 = \frac{\partial c^1}{\partial x}(0,0)$$

$$f_0^i = f^i(0,0), \quad f_1^1 = \frac{\partial f^1}{\partial w}(0,0)$$

where the notation (0,0) shows that the respective quantity is calculated at the critical point, where $x = 0$, $U_j = 0$.

Since f_0^1 and f_1^1 have different values at the right and left of the critical point, we shall consider Eq.(2.2) separately at the right and left of point $x = 0$.

Everywhere below, Latin indices run from 1 to n, and the Greek ones from 2 to n.

Expressing $\partial u_j/\partial t$ and $\partial u_j/\partial x$, respectively, in terms of $\partial w_k/\partial t$ and $\partial w_k/\partial x$ using formulas (2.1), we obtain instead of (2.3)

$$\frac{\partial w_\mu}{\partial t} + c_0^\mu \frac{\partial w_\mu}{\partial x} = f_0^\mu - c_0^\mu \sum_{j=1}^n l_{j1}^\mu r_{j1} w \frac{\partial w}{\partial x} \tag{2.4}$$

Retaining in Eqs.(2.4) only terms of order unity we can obtain a quasi-steady solution of these equations, neglecting time derivatives which are of order δ. Integrating (2.4) we obtain

$$w_\mu = a_\mu x + b_\mu w^2 + w_{\mu 0}(t) \tag{2.5}$$

$$a_\mu = \frac{f_0^\mu}{c_0^\mu}, \quad b_\mu = -\frac{1}{2}\sum_{j=1}^n l_{j1}^\mu r_{j1}$$

We use solutions (2.5) for transforming Eqs.(2.2). Retaining in (2.2) terms of order unity and $\sqrt{\delta}$ we obtain

$$\frac{\partial w}{\partial t} + [c_1^1 w + sx + rw^2 + \varphi(t)]\frac{\partial w}{\partial x} = f_0^1 + \chi w \tag{2.6}$$

$$s = c_x^1 + \sum_{\mu=2}^n c_\mu^1 a_\mu, \quad r = c_{11}^1 + \sum_{\mu=2}^n c_\mu^1 b_\mu$$

$$\varphi(t) = \sum_{\mu=2}^n c_\mu^1 w_{\mu 0}(t), \quad \chi = f_1^1 - f_0^1 \sum_{j=1}^n l_{j1}^1 r_{j1}$$

To reduce Eq.(2.6) to a simpler form we introduce the new variable

$$c = c_1^1 w + sx + rw^2 \tag{2.7}$$

Multiplying Eq.(2.6) by $c_1^1 + 2rw$ we obtain

$$\frac{\partial c}{\partial t} + [c + \varphi(t)]\frac{\partial c}{\partial x} = \gamma + \alpha c + s\varphi(t) \tag{2.8}$$

$$\gamma = f_0^1 c_1^1, \quad \alpha = f_1^1 + s + f_0^1\left(\frac{2r}{c_1^1} - \sum_{j=1}^n l_{i1}^1 r_{j1}\right)$$

It is assumed here that Eq.(2.8) holds for the left and right from the point $x = 0$.

Note that in many cases functions $w_{\mu 0}(t)$ and, consequently, also $\varphi(t)$ can be taken as equal zero, since these quantities are determined by the perturbations arriving in the critical point neighborhood and connected to the characteristic velocities $c^\mu \neq 0$. In any case, if $\varphi(t)$ and $d\varphi/dt$ are small, which is true for a wide number of problems, then by denoted $c + \varphi(t)$ by c and neglecting in the right-hand side of Eq.(2.8) the terms containing $\varphi(t)$ and $d\varphi/dt$, we obtain

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = \gamma + \alpha c \tag{2.9}$$

Equation (2.9) defines both, the unsteady and steady solutions in the neighborhood of the critical point $x=0$.

Let us, first, consider the steady solutions of that equation. Integrating (2.9) when $\partial c/\partial t=0$ we obtain

$$x=\frac{1}{2\gamma}c^2-\frac{\alpha}{2\gamma^2}c^3+\text{const} \tag{2.10}$$

This shows that, if the solution is to pass through the critical point $x=0, c=0$, it is necessary that the constant in formula (2.10) be zero and that the inequalities

$$\gamma_-<0,\ \gamma_+>0 \tag{2.11}$$

be satisfied.

The subscripts minus and plus denote quantities to the left and right of the critical point, respectively. Condition (2.11) is a refinement of the previously derived condition (1.4).

The pattern of integral curves is shown in Fig.1, where for definiteness we select $\alpha_-<0, \alpha_+>0$. The changed pattern of integral curves for other combinations of α on the left and right of the critical point is readily obtainable.

Using (2.10) in the case of solution passing through the critical point it is possible to calculate that the characteristics of Eq.(2.9) pass through half of the critical point neighborhood of dimension δ in a time of order $\sqrt{\delta}$, which was taken as the characteristic time of solution change. This is also the time of linear inversion of perturbation defined by Eq.(2.9), if its amplitude is of order $\sqrt{\delta}$ and the characteristic length is of order δ. Form of the steady solution (2.10) and the relation between c and w defined by formula (2.7) confirm the previous assumptions on the order of magnitude of w in the coordinate origin neighborhood. Obviously there exits a class of unsteady solutions of the same order of magnitudes. Equality (2.5) corroborate the previous assumption that the quantities $w_\mu(x,t)$ are of order δ, if $w_{\mu 0}(t)$ is also of order δ.

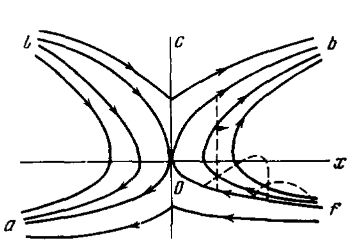

Fig.1

3. Let us compare Eq.(2.8) with the respective equation in /1/, obtained for the solution in the small neighborhood of the critical point when the right-hand sides of Eqs.(1.1) are continuous.

$$\frac{\partial c_*}{\partial t}+[c_*+\varphi(t)]\frac{\partial c_*}{\partial x}=\alpha c_*+\beta x+f(t) \tag{3.1}$$

$$c_*=c_1{}^1w+sx,\quad \alpha=f_1{}^1+s,\quad \beta=\Big(f_x{}^1+\sum_{\mu=2}^{n}f_\mu{}^1a_\mu\Big)c_1{}^1-f_1{}^1s$$

$$s=c_x{}^1+\sum_{\mu=2}^{n}c_\mu{}^1a_\mu,\quad f(t)=s\varphi(t)+c_1{}^1\sum_{\mu=2}^{n}f_\mu{}^1w_{\mu 0}(t)$$

The notation in the present paper is used in expressions for $c_*, \alpha, \beta, f(t)$, and expressions for s and $\varphi(t)$ are the same as in formula (2.6).

The comparison of Eqs.(2.8) and (3.1) shows that the cases of continuous and discontinuous function f^1, can be considered as one, if the relation between w and c is always defined by equality (2.7), and the equation for the determination of c in the δ-neighborhood of the critical point is of the form

$$\frac{\partial c}{\partial t}+[c+\varphi(t)]\frac{\partial c}{\partial x}=\gamma+\alpha c+\beta x+F(t) \tag{3.2}$$

$$F(t)=s\varphi(t)\ (\gamma\neq 0),\quad F(t)=f(t)\ (\gamma=0)$$

When $\gamma\neq 0$, the term βx is small in comparison with γ and αc, so that it can be neglected, and when $\gamma=0$, i.e. when function f^1 ($f_0{}^1=0$ is steady, it is possible to neglect in conformity with condition (1.3)), in formula (2.7) the term rw^2 which is small in comparison with terms proportional to w and x.

Function $\varphi(t)$ is the same in Eqs.(2.8) and (3.1) but, as was shown in Sect.2 it can be eliminated from Eq.(2.8), and as shown in /1/, functions $\varphi(t)$ and $f(t)$ can be eliminated from Eq.(3.1) by the introduction the new variables $c(x,t)-c^\circ(t)$ and $x(t)-x^\circ(t)$, where $c^\circ(t)$ and $x^\circ(t)$ are particular solutions of the system of differential equations

$$dc^\circ/dt=\alpha c^\circ+\beta x^\circ+f(t),\quad dx^\circ/dt=c^\circ+\varphi(t)$$

Thus by expressing the equation for c in the form (3.2) and the relation between c and

w in the form (2.7), we obtain the equation that defines the solution in the neighborhood of any critical point, and consider Eq.(3.2) under the conditions that

$$F(t) = \varphi(t) = 0 \tag{3.3}$$

The solution of Eq.(3.2) can be obtained by integrating along the characteristics of the system of equations

$$\frac{dc}{dt} = \gamma + \alpha c + \beta x + F(t), \ \frac{dx}{dt} = c + \varphi(t) \tag{3.4}$$

Equations (3.4) define unsteady and steady solutions. In the steady case system (3.4) with condition (3.3) yields for $c(x)$ a solution in the parametric form $c = c(t)$, $x = x(t)$.

In the case of appearance of solutions nonunique with respect to x it is necessary to introduce discontinuities, as in /2/, in order not to alter $\int c dx$, which was also done in /1/.

It is possible to obtain from Eq.(3.2) an equation for the determination of the small unsteady perturbations $c^*(x, t) = c(x, t) - C(x)$ of the steady solution $C(x)$

$$\frac{\partial c^*}{\partial t} + [C(x) + c^* + \varphi(t)]\frac{\partial c^*}{\partial x} = \left(\alpha - \frac{dC}{dx}\right)c^* + F(t) - \varphi(t)\frac{dC}{dx} \tag{3.5}$$

According to (2.7) the quantity c^* is connected to the quantity $w^* = w - W(x)$, where $W(x) = l_{j0}^1 U_j(x)$, by the relation

$$c^* = c_1^1 w^* + (2rW + w^*) w^* \tag{3.6}$$

where the second term is considerably smaller than the first.

The behavior of perturbations c^* and $w_\mu^* = w_\mu(x, t) - W_\mu(x)$, $W_\mu(x) = l_{j0}^\mu U_j(x)$ in the small neighborhood of point $x = 0$ is defined by Eq.(3.5) and formulas

$$w_\mu^* = b_\mu w^*(2W(x) + w^*) + w_{\mu 0}(t) \tag{3.7}$$

which follow from (2.5). Outside the neighborhood of point $x = 0$ it is possible to apply in conventional manner the linearized system of Eqs.(1.1), which we write for function w_j^* in the form

$$l_i^m(x, U_k) r_{ij}\left[\frac{\partial w_j^*}{\partial t} + C^m(x, U_k)\frac{\partial w_j^*}{\partial x}\right] = \chi_k^m(x) w_k^* \tag{3.8}$$

$$w_j^* = l_{jk}(0, 0) u_k^*, \quad u_k^* = r_{kj} w_j^*, \quad u_k^* = u_k(x, t) - U_k(x)$$

$$\chi_k^m = \left[\frac{\partial f^m}{\partial u_j} - \frac{\partial(l_i^m c^m)}{\partial u_j}\frac{dU_i}{dx}\right] r_{jk}$$

$$(i, j, k, m = 1, 2, \ldots, n)$$

Equations (3.8) are valid throughout the region of variation of x, for w_μ^* $(\mu = 2, \ldots, n)$, including the critical point neighborhood, but by virtue of the linearity of (3.8) the term containing w^{*2} which is negligibly small in the critical point neighborhood and outside it when $\gamma = 0$ $(f_0^1 = 0)$, is omitted in (3.7). The term containing w^{*2} must be added to the solution for w_μ^*, if the latter is to be made more accurate in the critical point neighborhood when $\gamma \neq 0$.

Equations (3.5) and (3.8) are necessary for solving the stability problem along the whole segment of x $(-L_1 \leqslant x \leqslant L_2)$, where a steady solution $U_j(x)$ of the system of Eqs.(1.1) exists and is considered. Depending on the problem formulation, the critical point $x = 0$ may coincide with right or left boundary of the x segment or be contained within it. For stability investigation of solution $U_j(x)$ along the whole segment it is necessary also to define the conditions of reflection of perturbations $w_j^*(x, t)$ from the boundary $x = -L_1$ or $x = L_2$, or from both simultaneously. As shown in /3/, the number of boundary conditions must correspond on the left and right to the number of positive and negative characteristic velocities of system (1.1), respectively.

We introduce here also, as in /1/, as the characteristic of behavior of small perturbation in the neighborhood of point $x = 0$, $c = 0$, the perturbation area

$$S = \int_{x_1}^{x_2} c^*(x, t)\, dx, \quad c^*(x, t) = c(x, t) - C(x) \tag{3.9}$$

Since the discontinuities introduced in the solution do not alter the perturbation area, and the evolution of solution is defined by Eqs.(3.4), the variation of perturbation area S with time is determined by the divergence of the vector composed from the right-hand sides of Eqs.(3.4)

$$\frac{dS}{dt} = S\left[\frac{\partial}{\partial x}(c + \varphi(t)) + \frac{\partial}{\partial c}(\gamma + \alpha c + \beta x + F(t))\right] = \alpha S \tag{3.10}$$

$$S(t) = S(0)\exp \alpha t$$

Formula (3.10) was obtained on the assumption that integration of (3.9) is carried out over the region outside which $c^*(x, t) = 0$.

If $c^*(x, t) \neq 0$ and also outside the integration interval (x_1, x_2), it is necessary to add to formula (3.10) the area increase q

$$\frac{dS}{dt} = \alpha S + q, \quad q(x, t) = \int\limits_{C(x)}^{C(x)+c^*(x,t)} c\,dc \tag{3.11}$$

$$q = q(x_2, t) - q(x_1, t) = \left[C(x_2)c^*(x_2, t) + \frac{1}{2}c^{*2}(x_2, t)\right] - \left[C(x_1)c^*(x_1, t) + \frac{1}{2}c^{*2}(x_1, t)\right]$$

The behavior of perturbations of steady solutions passing through singular points, i.e. for conditions $\gamma_\pm = 0$, $\alpha_+ = \alpha_-$, $\beta_+ = \beta_-$ was investigated in /1/ using Eqs.(3.4) and (3.3).

Here, we consider the behavior of perturbations of steady solutions passing through a critical point on the assumption that function f^1 is discontinuous. In that case in Eqs.(3.4) $\gamma_- < 0, \gamma_+ > 0, \alpha_- \neq \alpha_+$, while βx, $F(t)$ and $\varphi(t)$ are small in comparison with terms containing γ and c.

The pattern of integral curves for $\gamma \neq 0, \alpha_- \neq \alpha_+$ is shown in Fig.1. It suggests a deformed saddle. The direction of increase of t is indicated on curves by arrows.

In this case there are four solutions passing through the point $x = 0, c = 0$: aob, lof, lob and aof. Let us consider the development of perturbations in solution lof whose characteristics on solution lof converge at point $x = 0, c = 0$. Perturbations of this solution assume in conformity with (3.4) in time the shape of curvilinear triangle lying between curves fob or aol, depending on the perturbation sign, and bounded by the discontinuity, respectively, on the right or left (Fig.1).

In the absence of area increase q, the area of the curvilinear triangle lying to the right of the critical point $(c^* > 0)$ varies as $\exp \alpha_+ t$, while that of the triangle on the left of the critical point $(c^* < 0)$ varies as $\exp \alpha_- t$. The evolution in time $(t_1 < t_2 < t_3)$ shown in Fig.1 of the positive $(c^* > 0)$ perturbation of solution lof moving in the negative direction of axis x with $\alpha_+ > 0$. The forward front of that perturbation reaches the critical point in time $t \sim \sqrt{\delta_0}$, where δ_0 is the forward front coordinate at $t = 0$ and its rear front becomes a weak shock wave whose propagation velocity is the arithmetic mean of characteristic velocities ahead and behind the discontinuity. Since $\alpha_+ > 0$, the discontinuity begins to move at some instant of time in the positive direction of the x axis, as shown in Fig.1. When $\alpha_+ < 0$ the discontinuity moves in the direction of point $x = 0, c = 0$, which leads to the damping of perturbations.

The negative perturbation of solution lof moving along integral curve fo toward point 0 develops similarly, except that its forward front becomes a weak discontinuity, and in a finite time it assumes the shape of curvilinear triangle on the left of the critical point; from that instant its development is determined by the coefficient α_-. Depending on the sign of α_- the weak perturbation moves in the negative direction of the x axis $(\alpha_- > 0)$, which indicates an increase of perturbations, or in the positive direction of that axis x $(\alpha_- < 0)$ toward the critical point, which results in perturbation damping.

When the positive perturbation moves along the integral curve lo toward the critical point, its forward front, in this case, becomes a weak discontinuity. This perturbation is transformed in time in a curvilinear triangle on the right of the critical point, and its further evolution is determined by the sign of α_+.

A negative perturbation moving along the integral curve lo is transformed in time in a curvilinear triangle on the left of the critical point, and its increase of damping is determined by the sign of α_-.

We shall now consider the development of a positive perturbation of solution aob. Let at the initial instant of time $t = 0$ the amplitude of that perturbation at point $x = 0, c = 0$ be nonzero. Since any point of characteristic lo reaches point o in a finite time of order $\sqrt{\delta_0}$ and the amplitude of that perturbation is zero at the critical point. As the result, the whole perturbation leaves the δ-neighborhood of the critical point in time

$$t \sim \int_0^\delta \frac{dx}{\sqrt{x}} = 2\delta$$

The same occurs with the negative perturbation of solution aob.

It is possible to consider in the same way the development of positive and negative pertrubations of solutions lob and aof. Any initial perturbation of these solutions leaves the δ-neighborhood of the critical point in the finite time t of order $\sqrt{\delta}$, which substantially distinguishes solutions aob, lob and aof from solution lof, where the perturbations are always retained at the critical point.

The above analysis enables us to present cases in which the critical point coincides with the left or right end of the segment of x.

4. The investigation of perturbation development in Sect.3 and in /1/ was based on the assumption of no increase of perturbation area which may prove essential and affect the conclusions on the increase or decrease of perturbations in the critical point neighborhood. Let us show this.

Consider the interaction of perturbation $c^*(x, t)$ or $w^*(x, t) = w(x, t) - W(x)$ with perturbations propagating at velocities c^μ that are nonzero at the critical point. This enables us to estimate magnitude of area increase q in equality (3.11) which may represent a reflected signal generated by perturbations propagating at characteristic velocities c^μ.

The most interesting case is when these perturbations are themselves generated by perturbation $w^*(x, t)$. Because of this, we assume below that the order of magnitude of these perturbations is equal to their variations in the critical point neighborhood, with the characteristic time of variation of these perturbations is the same as that of $w^*(x, t)$.

Let us determine the behavior of quantities w_μ^* more exactly than in equality (3.7). In the latter the first term in the right-hand side contains a perturbation expressed in terms of w^*, and consequently, propagating at velocity c, only the second term can correspond to perturbations propagating at other characteristic velocities. To refine formula (3.7) we revert to the system of Eqs.(1.1) and consider these equations for i from 2 to n, expressing u_j in terms of w_k in conformity with (2.1)

$$(\delta_k^\mu + a_k^\mu)\left(\frac{\partial w_k}{\partial t} + c^\mu \frac{\partial w_k}{\partial x}\right) = f^\mu, \quad \delta_k^\mu + a_k^\mu = l_j^\mu r_{jk} \tag{4.1}$$

where the Kronecker delta $\delta_k^\mu = 1$ when $\mu = k$ and $\delta_k^\mu = 0$ when $\mu \neq k$, and functions a_k^μ depend on w_i and x and vanish at the critical point. We solve these equations for derivatives of w_μ with respect to x, and obtain

$$\frac{\partial w_\mu}{\partial x} = -\frac{1}{c^\mu}\frac{\partial w_\mu}{\partial t} + A_{\mu\nu}\frac{\partial w_\nu}{\partial t} + F_\mu + B_\mu \frac{\partial w}{\partial t} + D_\mu \frac{\partial w}{\partial x} \tag{4.2}$$

where $A_{\mu\nu}, B_\mu, D_\mu$ are functions of w_k and x, which vanish at zero values of arguments, and functions F_μ are linearly expressed in terms of f^μ.

We expand all these functions in the neighborhood of the critical point in series in w_k and x. We determine dw/dt and v_ν using, respectively, formulas (2.6) and (2.5), and substitute these in the right-hand sides of formula (4.2), neglecting the terms $w_{\mu 0}(t)$, which, as shown below, on the assumptions made at the beginning of this Section, give corrections of a higher order of smallness than the remaining terms in (2.5) and (2.6). Restricting the obtained expression to a few first terms of expansions, we obtain

$$\frac{\partial w_\mu}{\partial x} = a_\mu + b_\mu w + e_\mu x + (g_\mu w + h_\mu w^2 + k_\mu x)\frac{\partial w}{\partial x} \tag{4.3}$$

where $a_\mu, b_\mu, e_\mu, g_\mu, h_\mu, k_\mu$ are constant coefficients of the expansion.

Using integration by parts, we obtain for w_μ the more accurate formula

$$w_\mu = a_\mu x + \frac{1}{2} e_\mu x^2 + (b_\mu - k_\mu)\int_0^x W(x)\,dx + g_\mu \frac{w^2}{2} + h_\mu \frac{w^3}{3} + k_\mu x w + v_\mu \tag{4.4}$$

$$v_\mu = (b_\mu - k_\mu)\int_0^x w^*\,dx + p_\mu(t)$$

where $(p_\mu(t)$ is a derivative function dependent only on time.

This equality remains valid when function w has discontinuities, since integration over a discontinuity can be carried out taking dw/dx as a generalized function. This statement is based on that the variation of quantities at a discontinuity coincides with an accuracy to the third order of smallness with respect to the discontinuity amplitude corresponds with their variation in some simple wave /2/ that approximates the discontinuity at the given instant of time.

The terms neglected in equality (4.3) result in an error of order δ^2, while in equality (4.4) the error is of order δ^3, if $\gamma = 0$, when w is of order δ and dw/dx is of order unity. When w and dw/dx if $\gamma \neq 0$ are, respectively, of order $\delta^{1/2}$ and $\delta^{-1/2}$, the error in equality (4.3) due to neglected terms is of order δ, and in (4.4) it is of order δ^2.

All terms in the right-hand side of equality (4.4) except the latter are expressed in terms of x and w at the running point, hence they relate to the stationary background or to a perturbation moving at velocity c.

The expression v_μ, which at the considered point is not explicitly dependent on w^*, represents a perturbation that corresponds to other characteristic velocities, basically to the characteristic velocity c^μ.

In conformity with the assumption made at the beginning of this point, we shall consider that the order of magnitude of v_μ is determined by the first integral term. Note that v_μ provides a more exact definition of the term $w_{\mu 0}(t)$ in equality (2.5). We can verify the validity of neglecting terms $w_{\mu 0}(t)$ in the derivation of Eqs.(4.3) when $\gamma = 0$, as well as when $\gamma \neq 0$, by assuming that v_μ and $w_{\mu 0}(t)$ are quantities of the same order.

Equality (4.4) shows that perturbation $w^*(x, t)$ in the critical point neighborhood generates perturbations v_μ that correspond to other characteristic velocities, and are equal in the order of magnitude to the area S calculated by formula (3.9).

Assuming that the critical point lies at a finite distance from the segment boundaries, where the conditions of waves reflection are specified with finite coefficients, and the interconvertibility of waves inside the segment cannot materially affect their order of magnitude, it can be readily proved that the area increase q is determined by v_μ.

This implies that the order of magnitude of q is $\alpha_1 S$, where α_1 is a finite number, independent of δ.

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